Normal numbers and Selection rules

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Let $M = (\Sigma, \psi, \sigma_0, F)$ be a countable automaton over $\{0,1\}$. That is, Σ is a countable set, $\sigma_0 \in \Sigma$, $F \subset \Sigma$ and $\psi : \Sigma \times \{0,1\} \to \Sigma$. Let $\beta = \beta(0)\beta(1)\beta(2)\cdots \in \{0,1\}^{\mathbb{N}}$ be an infinite binary word, where $\mathbb{N} = \{0,1,2,\cdots\}$. Then, M defines a way of choosing a subword (*selection rule*) by deciding whether $\beta(i)$ should be included in it or not using the information $\beta[0,i)$, where for $\beta \in \{0,1\}^{\mathbb{N}}$ and $i \le j$, denote $\beta[i,j) = \beta(i)\beta(i+1)\cdots\beta(j-1)$. That is, let $S = \{s_0 < s_1 < \cdots\} \subset \mathbb{N}$ be such that $S = \{i \in \mathbb{N};\ \psi(\sigma_0,\ \beta[0,i)) \in F\}$, where

$$\psi(\sigma_0,\ \beta[0,i)) = \psi(\cdots\psi(\psi(\sigma_0,\beta(0)),\beta(1))\cdots,\beta(i-1)).$$

We define a subword of β, denoting $\beta[S] \in \{0,1\}^{\mathbb{N}}$ by $\beta[S](n) = \beta(s_n)$ $(\forall n \in \mathbb{N})$ if S is an infinite set. We shall denote this S by $S(M,\beta)$.

We look for the condition on the selection rule M so that the subsequence $\alpha[S(M,\alpha)]$ is a binary normal number provided that α is a binary normal number and $S(M,\alpha)$ has a positive lower density. In this case, we say that the selection rule M *preserves normality.*

The following theorem, which is originally proved by T. Kamae and B. Weiss {2} (there is a nontrivial error in the proof which should be corrected) as a generalization of the case of finite automata (Y. N. Agafanov {3}).

Theorem 1. ({2}) *For a normal number $\alpha \in \{0,1\}^{\mathbb{N}}$ and a countable automaton M, if F is a finite set such that the subset $S(M,\alpha)$ of $\mathbb{N}$ has a positive lower density, then $\alpha[S(M,\alpha)]$ is a normal number. That is, the selection rule M preserves normality if $\#F < \infty$. .*

In the case that the selection rule M uses only the information i included in $\beta(0)\beta(1)\cdots\beta(i-1)$, $S(M,\beta)$ becomes indifferent to β, so that we may assume

$$S(M,\beta) = \{i;\ \gamma(i) = 1\} \ \ \text{(indifferent to } \beta) \qquad (*)$$

with $\gamma \in \{0,1\}^{\mathbb{N}}$. We say in {1} that γ is *completely deterministic*, if any pseudo-generic measure of γ with respect to the shift on $\{0,1\}^{\mathbb{N}}$ has the Kolmogorov-Sinai's entropy 0.

Theorem 2. ({1}) *Let the selection rule defined in $(*)$ satisfy that the set $\{i \in \mathbb{N};\ \gamma(i) = 1\}$ has a positive lower density. Then, it preserve normality if and only if γ is completely deterministic.*

In this talk, we generalize these theorems.

References:

{1} Israel J. Math 16 (1968), 121-149. {2} Israel J. Math 21 (1973), 101-110.
{3} Dokl. Akad. Nauk SSSR 179 (1968).